CSC165H1: Mathematical Expression and Reasoning for Computer Science **Fall 2018**

Lecture 9: October 9

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9.1 More on Simple Induction

The goal of simple induction is to prove statements of the following form:

$$\forall n \in \mathbb{N}, P(n)$$

and variants of it, where $P(n)$ is only true after some point, m:

$$\forall n \in \mathbb{N}, n \geq m \Rightarrow P(n), \text{ where } m \in \mathbb{N}$$

When the base case ($P(0)$) and the induction step ($P(n) \Rightarrow P(n+1)$) is proven, we have essentially shown that $P(n)$ is true for all natural n's.

9.1.1 Example

Consider the statement:

"The sum of the first n odd natural numbers is a perfect square."

In other words, we are showing:

$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} 2i - 1 = n^2$$

The base case for $P(0)$ can be shown trivially.

Now, we assume that $P(n)$ and show $P(n+1)$.

$$\begin{aligned}
\sum_{i=1}^{k=1} 2i - 1 &= 1 + 3 + 5 + \cdots + (2k-1) + (2(k+1) - 1) \\
&= \sum_{i=1}^{k} 2i - 1 + (2(k+1) - 1) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

Therefore, we have proven the statement.

9.1.2 Another Example

Prove that:

$$\forall x, y \in \mathbb{Z}, \forall n \in \mathbb{N}, 5|(x-y) \Rightarrow 5|\,(x^n - y^n)$$

For some arbitrary integers x, y, we can prove by induction on n.

Base Case: Prove $P(0)$.

When $n = 0$, we know that $5|(x^n - y^n)$ since $x^n - y^n = 0$.

Induction Step: Let $k \in \mathbb{N}$, and assume $P(n)$. That is:

$$5 \mid (x-y) \Rightarrow 5 \mid \left(x^k - y^k\right)$$

and we want to prove:

$$5 \mid (x-y) \Rightarrow 5 \mid \left(x^{k+1} - y^{k+1}\right)$$

Let $x - y = 5c$ for some $c \in \mathbb{Z}$. Then, by the inductive hypothesis, we know $5|(x^k - y^k)$.

$$\begin{aligned} x^{k+1} - y^{k+1} &= x \cdot x^k - y \cdot y^k \\ &= x\left(x^k - y^k\right) + xy^k - yy^k \\ &= x\left(x^k - y^k\right) - (x-y)y^k \\ &= x(5d) + 5(cy^k) \\ &= 5(xd + cy^k) \\ &= 5e \end{aligned}$$

Therefore, we can conclude that $5|(x^k - y^k)$.