

Lecture Mar 6

Wednesday, March 6, 2019 11:10 AM

Recursive Binary Search (A, f, l, x):

```
if  $f = l$  then  
  if  $A[f] = x$  then return  $f$   
  else return 0
```

```
 $m \leftarrow \lfloor \frac{f+l}{2} \rfloor$   
if  $A[m] \geq x$  then Recursive Binary Search ( $A, f, m, x$ )  
else Recursive Binary Search ( $A, m+1, l, x$ )
```

Let $B: \mathbb{Z}^+ \rightarrow \mathbb{N}$ be such that $B(n)$ is the worst case number

of comparisons with x performed by Recursive Binary Search (A, f, l, x)
where $n = l - f + 1$

From the code, $B(1) = 1$ and $B(n) = 1 + \max \{ B(\lfloor \frac{n}{2} \rfloor), B(\lceil \frac{n}{2} \rceil) \}$

$$\begin{aligned}\bullet \quad n-f+1 &= \left\lfloor \frac{f+l}{2} \right\rfloor - f + 1 \\ &= \left\lfloor \frac{l+f-2f}{2} \right\rfloor + 1 = \left\lfloor \frac{l-f}{2} \right\rfloor + 1 = \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil\end{aligned}$$

$$\bullet \text{ For second part, by def of } n, \quad n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$$

Then, since $B(n)$ non-decreasing. $B(\lceil \frac{n}{2} \rceil) \geq B(\lfloor \frac{n}{2} \rfloor)$

$$\begin{aligned}B(n) &= 1 + \max \{ B(\lceil \frac{n}{2} \rceil), B(\lfloor \frac{n}{2} \rfloor) \} \\ &= 1 + B(\lceil \frac{n}{2} \rceil)\end{aligned}$$

We can then use Master Theorem.

Insertion Sort (A):

```
1   $i \leftarrow 1$   
2  while  $i \leq \text{length}(A)$  do  
3     $j \leftarrow i$   
4    while  $j > 1$  AND  $A[i] < A[j-1]$  do  
5       $A[j] = A[j-1]$ 
```

```

4     while  $j > 1$  AND  $A[i] < B[j-1]$  do
5          $B[j] = B[j-1]$ 
6          $i \leftarrow j-1$ 
7          $B[j] \leftarrow A[i]$ 
8          $i \leftarrow i+1$ 
9     return (B)

```

$$\text{Lemma 1: } T_{JS}(n) \leq 2n^2 + 4n + 2 \in O(n^2)$$

Proof. Let $n \in \mathbb{N}$ be arbitrary, A be an arbitrary array of length n . There are exactly n complete iterations of the outer while-loop. Each iteration consists of 4 steps (L2,3,7,8) and an execution of the inner while-loop.

Each inner for-loop has at most 4 operations (L4,5,6).

During iteration i of the outer loop, at most $(n-i)$ complete iterations of the inner while-loop are performed.

The final (incomplete) iteration takes at most 2 steps.

Hence, total is at most,

$$\sum_{i=1}^n [4(i-1) + 2 + 4] = 2n^2 + 4n$$

The final (incomplete) iteration of the outer loop takes 1 step.

Added to L1, final total is:

$$2n^2 + 4n + 2$$

$$\text{Lemma 2: } T_{JS}(n) = \Omega(2^n)$$

Let $n \in \mathbb{N}$ be arbitrary. Consider input $A = [n, n-1, \dots, 1]$

Claim: For $1 \leq i \leq n$, during iteration i of the outer while-loop $(4i+1)$ steps are performed.

After iteration i , $j=i$ and the first i elements of B are $[n-i+1, \dots, n]$

During the first iteration, 5 steps are performed ($l, 2, 3, 4, 7, 8$)

Afterwards, $j=1$, and the first element of B is $A[i] = n$.

If $i < n$, then during the next $(i+1)^{st}$ iteration of the outer while-loop
4 steps are performed in addition to the inner while-loop. $A[i+1] = n-1$

There are i complete iterations of the inner while-loop each consisting of 4 steps

In the last (incomplete) iteration of the inner while-loop only 1 step is performed

So $4i+5$ steps are performed on the $(i+1)^{st}$ iteration of the outer while-loop.

The final (incomplete) iteration of the outer while loop takes 1 step

Also, there is 1 step before the loop.

$$\text{Hence: } T_{IS}(n) = 4 + 2 + \sum_{i=1}^{n-1} (4i+5) = 2n^2 + 3n + 1$$

Since n is arbitrary, $T_{IS}(n) \geq 2n^2 + 3n + 1$ for all $n \geq 1$.

Hence $T(n) \in \Omega(n^2)$.

Algorithm is correct: satisfies specifications

Preconditions: statement involving variable used in the algorithm
Before execution begins it can describe which inputs are allowable.

Postconditions: statement involving variables used
Contains facts that must be true when algorithm ends.
Often, it describes the correct output.

Termination

Partial Correctness: if eventually terminates the post condition holds.

Total Correctness: termination + partial.