

Lecture Mar 25

Monday, March 25, 2019 11:11 AM

Claim. The language $L = \{a^i b^j \mid i, j \in \mathbb{N}\}$ is not accepted by any finite automaton.

Proof. Suppose $L = \delta(M)$ where $M = (Q, \{a, b\}, \delta, q_0, F)$ is a DFA

Let $n = |Q|$. Consider the states $q_i = \delta^*(q_0, a^i)$ for $i = 0, \dots, n$.

By pigeonhole principle, there exists $0 \leq i < j \leq n$ such that $q_i = q_j$

Then

$$\begin{aligned} \delta^*(q_0, a^j b^j) &= \delta^*(\delta^*(q_0, a^i), b^j) \\ &= \delta^*(\delta^*(q_0, a^j), b^j) \quad \text{since } q_i = q_j \\ &= \delta^*(q_0, a^j b^j) \in F. \end{aligned}$$

Since $a^j b^j \in L = \delta(M)$, but then $a^i b^j \notin \delta(M)$.

This contradicts the assumption that $L = \delta(M)$ since $a^i b^j \notin L$.

Hence $L \neq \delta(M)$ ■

Theorem (Pumping Lemma). For every FA $M = (Q, \Sigma, \delta, q_0, F)$ there exists $n \in \mathbb{Z}^+$:

if $|x| \geq n$ then exists $u, v, w \in \Sigma^*$ such that:

- $v \neq \lambda$
- $|uv| \leq n$
- $x = uvw$
- $\forall k \in \mathbb{N} \quad uv^k w \in \delta(M)$

Proof. By subset construction, assume M is a DFA.

Let $n = |Q|$

Let $x \in \delta(M)$ be an arbitrary such that $|x| \geq n$.

Say $x = x_1 \dots x_m$ where $x_1 \dots x_m \in \Sigma$ and $m \geq n$. (Consider the first n letters)

Consider the states $q_i = \delta^*(q_0, x_1 \dots x_i)$ for $i = 0, \dots, n$

By pigeonhole principle there exists $0 \leq i < j \leq n$ such that $q_i = q_j$.

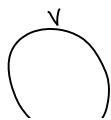
Let $u = x_1 \dots x_i$

$v = x_{i+1} \dots x_j$ Then $x = uvw$.

$w = x_{j+1} \dots x_m$

Since $i < j$, $v \neq \lambda$. Since $j \leq n$, $|uv| \leq n$.

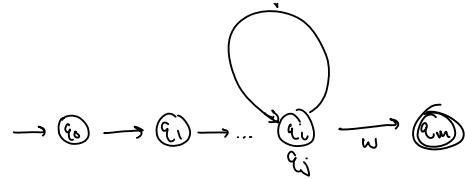
$\delta^*(q_0, uv^k w) = q_i$ for all $k \geq 0$



Since $v < j$, $v \neq \lambda$. Since $j \leq n$, $|uvw| \leq n$.

$\delta^*(q_0, wv^k) = q_j$ for all $k \geq 0$.

$$\begin{aligned} \text{So, } \delta^*(q_0, wv^k w) &= \delta^*(\delta^*(q_0, wv^k), w) \\ &= \delta^*(q_j, w) \\ &= \delta^*(\delta^*(q_0, vw), w) \\ &= \delta^*(q_0, ww) = \delta^*(q_0, x) \in F. \end{aligned}$$



Since $x \in J(M)$, $wv^k w \in J(M)$.

Application of Pumping Lemma

$$PAL = \{ z \in \{0,1\}^* \mid z^{rev} = z \} \neq J(M) \text{ for any FA } M.$$

Proof: Suppose $PAL = J(M)$ for some FA M .

Then, by PL, $\exists n \in \mathbb{Z}^+ \forall x \in PAL [|x| \geq n \text{ IMP. } (\exists u, v, w \in \Sigma^* \text{ such that } x = uvw \text{ AND } |uv| \leq n \text{ AND } v \neq \lambda \text{ AND } \forall k \in \mathbb{N}, uw^k w \in PAL)]$

Consider $x = 0^n \mid 0^n \in PAL$.

Then $x = uwv$ where $|uv| \leq n$, $v \neq \lambda$, $\forall k \in \mathbb{N}$, $uw^k w \in PAL$.

(u is first i letters, v is last j letters) $u = 0^i$ $v = 0^j$ for some $i \geq 0, j \geq 1$ $(v \neq \lambda)$
 $|uv| \leq n$, $|u|, |v| \leq n$ such that $i+j \leq n$

But $uv^2 w = 0^i 0^j 0^j w = 0^j uwv = 0^{j+n} \mid 0^n \notin PAL$. (Contradiction).

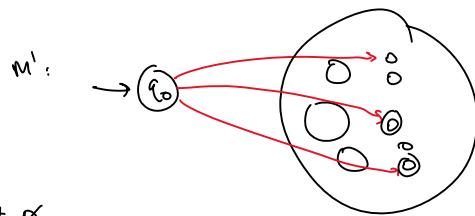
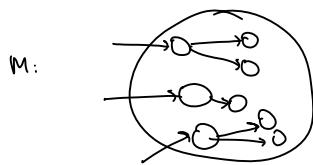
There is a contradiction, hence $PAL \neq J(M)$ for any FA M .

Variants of NFA

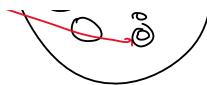
- NFA with multiple start states, $M = (Q, \Sigma, \delta, Q_0, F)$, where $Q_0 \subseteq Q$.

If $L = J(M)$ for some NFA with multiple start states, there exists NFA M' such that $J(M) = J(M')$.

Proof: Consider first letter Create arbitrary start state



and also $q_0 \in F$ iff $Q_0 \cap F \neq \emptyset$.



and also $q_0 \in F$ iff $Q_0 \cap F \neq \emptyset$.

$M' = (Q \cup \{q_0\}, \Sigma, \delta', q_0, F')$ where $q_0 \notin Q$.

For all $a \in \Sigma$, $\delta'(q_0, a) = \bigcup \{\delta(q, a) \mid q \in Q_0\}$

$\delta'(q_0, a) = \delta(q, a)$ for all $q \in Q$.

$F' = F$ if $Q_0 \cap F = \emptyset$. otherwise $F' = Q_0 \cup \{q_0\}$.

Next, need to prove $J(M') = J(M)$.

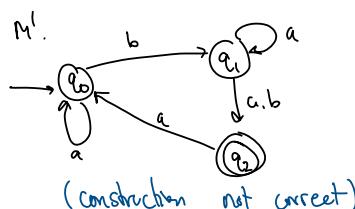
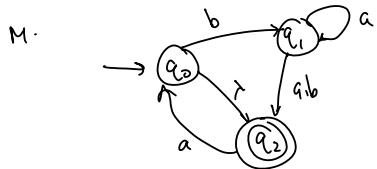
- NFA with λ transitions $M = (Q, \Sigma, \delta, q_0, F)$, $\delta = Q \times (\Sigma \cup \{\lambda\}) \rightarrow P(Q)$.

$x \in J(M)$ iff there is a path from q_0 to a state in F .
such x is concat. of edges on that path

$M' = (Q, \Sigma, \delta', q_0, F)$ For every state $q \in Q$. let

$E(q) = \{q' \mid q' \text{ can be reached by edge labeled by } \lambda\}$
 $\lambda \in E(q)$

$$\delta'(q', a) = \bigcup \{E(q') \mid q' \in \delta(q, a)\}$$



Instead, consider $M' = (Q, \Sigma, \delta', E(q_0), F)$ (multiple start states)