

Lecture Feb 6

Wednesday, February 6, 2019 11:11 AM

Strong Induction Variant for Tuples

Want to prove $\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, p(m, n)$

Let $Q(n) : \forall m \in \mathbb{N} \ p(m, n)$.

Let $m \in \mathbb{N}$ be arbitrary.

Assume $Q(i)$ for all $0 \leq i \leq m$.

Let $n \in \mathbb{N}$ be arbitrary.

Assume $p(m, j)$ for $0 \leq j \leq n$

:

$p(m, n)$

$\forall n \in \mathbb{N}. \ p(m, n)$

$Q(m)$

$\forall m \in \mathbb{N}. \ Q(m)$.

Let S be a recursively defined set

Let $p : S \rightarrow \{T, F\}$

Base Case : $p(s)$

Induction Step: Assume $p(s')$ for all components of s .

Prove $p(s)$

$\forall s \in S. \ p(s)$.

Functions on Recursively Defined Sets

Ex 1 M : monotone propositional formulas

(refer to notes Feb 4) $N_v(f), N_c(f)$

Ex 2 Arithmetic expressions

Basic Cases: $\mathbb{N} \subseteq A$

Ex 2 Arithmetic Expressions

Base Cases: $N \subseteq A$

Constructor Cases: $a, b \in A$ then $ab, a+b \in A$.

Let $\text{val}: A \rightarrow \mathbb{N}$ be such that

$$\begin{cases} \text{val}(n) = n & \text{for all } n \in \mathbb{N} \\ \text{val}(a+b) = \text{val}(a) + \text{val}(b) \\ \text{val}(a \cdot b) = \text{val}(a) \cdot \text{val}(b) \end{cases}$$

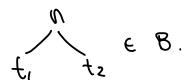
$$\begin{aligned} \text{val}(3 \times 2 + 1) &= 9 && \text{if } 2+1 \text{ evaluated first} \\ &= 7 && \text{if } 3 \times 2 \text{ evaluated first.} \end{aligned}$$

The set A is well-defined, but \mathbb{A} is not.

Ex 3 Let B denote the set of all binary trees

Base Case: The empty tree is in B

Constructor: If $t_1, t_2 \in B$, n is a node, then



For some root, $\text{left}(t)$, $\text{right}(t)$ refers to left and right subtree.

Define $N: B \rightarrow \mathbb{N}$ $N(t)$ denotes the number of nodes in t

$$\begin{cases} N(\text{empty tree}) = 0 \\ N(t) = N(\text{left}(t)) + N(\text{right}(t)) + 1. \end{cases}$$

Define $L: B \rightarrow \mathbb{N}$ to denote the number of leaves.

$$\begin{cases} L(\text{empty tree}) = 0 \\ L(\text{one node}) = 1 \\ L(t) = L(\text{left}(t)) + L(\text{right}(t)). \end{cases}$$

Theorem A binary tree of size n has at most $\lceil n/2 \rceil$ leaves.

$$\forall b \in B, L(b) \leq \lceil N(b)/2 \rceil.$$

Strong Induction

Let $n \in \mathbb{N}$ be arbitrary
Suppose $\forall j \in \mathbb{N} (j \leq n \implies p(j))$

Let $b \in B$ be arbitrary.

Suppose $N(b) = n$

Structural Induction

Let $b \in B$ be arbitrary
Base.

$q(\text{empty}) \rightarrow \textcircled{1}$
 $q(\text{one node})$

Let $b \in B$ be arbitrary.
 Suppose $N(b) = n$
 $L(b) \leq \lceil N(b)/2 \rceil$
 $N(b) = n \text{ IMP } L(b) \leq \lceil N(b)/2 \rceil$
 $\phi(n)$
 $\forall n \in \mathbb{N}, \phi(n)$ by induction.

$q(\text{empty}) \rightarrow \textcircled{1}$
 $q(\text{one node})$
 Constructor:
 Suppose $q(\text{left}(b))$ and $q(\text{right}(b))$
 $g(b) \rightarrow \textcircled{2}$

① For empty tree, $N(b) = L(b) = 0$.
 For one node tree, $N(b) = L(b) = 1$. \Rightarrow Both are true

② By definition:

$$N(b) = N(\text{left}(b)) + N(\text{right}(b)) + 1$$

$$L(b) = L(\text{left}(b)) + L(\text{right}(b))$$

By inductive hypothesis:

$$L(\text{left}(b)) \leq \lceil N(\text{left}(b))/2 \rceil$$

$$L(\text{right}(b)) \leq \lceil N(\text{right}(b))/2 \rceil$$

$$\begin{aligned} \text{Then: } L(b) &= L(\text{left}(b)) + L(\text{right}(b)) \\ &\leq \lceil \frac{N(\text{left}(b))}{2} \rceil + \lceil \frac{N(\text{right}(b))}{2} \rceil \end{aligned} \quad (\text{by hypothesis})$$

$$\leq \lceil \frac{N(\text{left}(b)) + N(\text{right}(b)) + 1}{2} \rceil$$

Need to prove this $= \lceil \frac{N(b)}{2} \rceil$