

Lecture Feb 11

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Structural Induction

Let $p: S \rightarrow \{T, F\}$ be recursively defined predicate

Let $s \in S$ be arbitrary

If s is base case:

.....
 $p(s)$

If s is a constructor case

Assume $p(s')$ is true for all parts of s

.....
 $p(s)$

$\forall s \in S, p(s)$ structural induction

Structural Induction Justification

Let $E_0 =$ subset of S in the base case

$E_i =$ subset obtained from E_0 by applying constructor i times

Then, structural induction is strong induction on the size of elements in S .

Induction Proofs as Contradiction

Consider smallest counterexamples. Prove it can't exist or there is a smaller counterexample.

Example Every integer greater than 1 can be written as product of primes.

Proof: Suppose the claim is false.

Let n be the smallest integer > 1 that cannot be written as product of primes.

If n is prime, then it can be written as a product. Hence, n is composite.

Hence, $\exists k, m \in \mathbb{Z}, k > 1, m > 1$ and $n = km$.

But $km < n$ so they can be written as product of primes

Contradiction is introduced.

Hence, claim is true.

Well-Ordering Principle

Def: A set S is partially ordered if there exists a binary predicate.

$R: S \times S \rightarrow \{T, F\}$

nil. Hint for all $x, y \in S$.

Def: A set S is partially ordered if there exists a binary predicate.

$$R: S \times S \rightarrow \{T, F\}.$$

such that for all $x, y, z \in S$:

1. reflexivity: $R(x, x) = T$
2. asymmetry: $R(x, y)$ AND $R(y, x)$ IMPLIES $x = y$
3. transitivity: $R(x, y)$ AND $R(y, z)$ IMPLIES $R(x, z)$

Examples: (\mathbb{Z}, \leq) , $(P(\{1, 2, 3\}), \subseteq)$ are partially ordered.

Non-Example: For $S = \mathbb{C}$. $R(x, y) = "|x| \leq |y|"$.
Since $|1| = |i| = 1$ but $1 \neq i$.

Def: S is totally ordered if there exists a partial order R
such that: for all $x, y \in S$.

$$\text{Compatibility} \quad R(x, y) \text{ OR } R(y, x)$$

Example: (\mathbb{Z}, \leq) is a total order

$(P(\{1, 2, 3\}), \subseteq)$ is not. Since $\{1\}, \{3\}$ are neither greater or lesser.

Def: A totally ordered (or partially ordered) set (S, R) is well ordered
if every non-empty subset of S has a smallest element (wrt R).

Example: (\mathbb{N}, \leq) is well-ordered.

(\mathbb{Z}, \leq) is not well-ordered.

but consider $R(x, y) = (|x| < |y|)$ OR $(|x| = |y| \text{ AND } x < y)$.

then (\mathbb{Z}, R) is well-ordered

(\mathbb{Q}, R_2) is well ordered when (and totally ordered)

$$R_2\left(\frac{p}{q}, \frac{r}{s}\right) = (\max(p, q) < \max(r, s)) \text{ OR}$$

$$(\max(p, q) = \max(r, s) \text{ AND } \frac{p}{q} \leq \frac{r}{s}).$$

Induction with WOP

To prove $\forall e \in S, P(e)$ where (S, \leq) is any well-ordered set.

1. To obtain contradiction, suppose $\forall e \in S, P(e)$ false.
2. Let $C = \{e \in S \mid P(e) \text{ is false}\}$ be the set of counterexamples.
3. $C \neq \emptyset$ by def in 1.2.
4. Let e be smallest element in C exists by WOP
5. Let e' = smaller counterexample.
6. $e' \in C$

- i
- 6 $e' \in C$
- 7 $e' < e$
- 8 Contradiction $4 \sim 7$
- 9. $\forall e \in S. p(e)$